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ABSTRACT

Twenty primary school students were randomly assigned by a matching procedure to either an experimental or control group. During the whole school year, both groups were taught mathematics according to the curriculum in the school; however, the groups followed different programs in fractions. Students in the experimental curriculum were provided with opportunities to develop number sense in a program where class discussion and the number line takes a central place as a mental model for formal reasoning in learning fractions. By contrast, in the curriculum of the control group the circle was used as a model to reach formalization. At various time points, the teaching and learning processes and the outcomes of the two curricula were investigated by participant-observations, in-depth interviews and tests for mathematical understanding. This comparison particularly shows qualitative differences in operating with fractions between experimental group and control group. In their mental models, strategies and reflections students from the experimental group demonstrated clearly more insight i.e., aspects of number sense in processing fractions, both relatively when compared to the controls, as absolutely in regard to the national standards. Moreover, we observed differential effects in comparing the experimental with the control group. The outcomes of the study are discussed and some implications for curriculum theory and practice are presented. (Contains 65 references. (Author/CCM)

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Learning for mathematical insight: a longitudinal comparison of two Dutch curricula on fractions

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Abstract
Twenty primary school students were randomly assigned by a matching procedure to either an experimental or control group. During the whole school year, both groups were taught mathematics according to the curriculum in the school, however the groups followed different programs in fractions. Students in the experimental curriculum were provided with opportunities to develop number sense in a program where class discussion and the number line takes a central place as a mental model for formal reasoning in learning fractions. By contrast, in the curriculum of the control group the circle was used as a modél to reach formalization. At various time points, the teaching and learning processes and the outcomes of the two curricula was investigated by participant-observations, in-depth interviews and tests for mathematical understanding. This comparison particularly shows qualitative differences in operating with fractions between experimental group and control group. In their mental models, strategies and reflections students from the experimental group demonstrated clearly more insight i.e., aspects of number sense in processing fractions, both relatively when compared to the controls, as absolutely in regard to the national standards. Moreover we observed differential effects in comparing experimental and control group. The outcomes of the study are discussed and some implications for curriculum theory and practice are presented.

1. Introduction

Mathematical insight is widely recognized as an important goal of education (Van Hiele, 1986; Perkins & Unger, 1999; Reigeluth, 1999). However, it is known from many studies that students have difficulty in applying their mathematical knowledge in meaningful ways in mathematical modeling and formal mathematics. For example, several years ago one of the Dutch national newspapers devoted an article in its educational supplement to the subject of fractions in primary schools. The headline 'Fractions happen to be difficult' summarized the thoughts of many. Many educators and researchers confirm the problems which students encounter in learning fractions (Behr, Lesh, Post & Silver, 1983; Carraher & Schliemann, 1991; Hasemann, 1981; Kamii & Clark, 1995; Graeber & Tanenhaus, 1993), especially when fractions and operations with fractions are not firmly connected to concrete experiences (Hart, 1981; Hiebert, 1988) or significant situations (Streefland, 1982, 1987 & 1990). Behr et al. (1983) attempt to seek the cause of students' difficulties in learning fractions in the necessary transition from concrete experiences to formal reasoning, and in the representation model of fractions, as well as in the many subsidiary concepts needed to obtain proficiency in fractions.

In exploring the question of how to facilitate the process of transition from concrete experiences to formal reasoning, several fraction-generating activities could be mentioned. In this paper we will focus on two of these activities.
Firstly, Streefland proposed fair-sharing as an activity to generate fractions by using the circle, and later the ratio-table, as models. Sharing three pizzas among four children can be considered as prototype of this kind of situation. Streefland showed how fair sharing stimulates the development of a 'language of fractions'. By considering equivalent situations of fair sharing equivalent fractions emerge and a road is paved toward formal reasoning with fractions. However, as equivalent fractions are mainly seen as equivalent situations, fractions as *objects* are not clearly visualized. Though Streefland's ideas match with opinions held only a few years ago, a number of developments ask for further consideration of teaching fractions.
Secondly, there is a strong tendency to consider learning fractions from the viewpoint of acquiring number sense (Mcintosh, Reys and Reys, 1992). There is some evidence that using a bar as a model and a number line as abstraction of the bar can provide for such a curriculum (Keijzer, 1997).

Although the two approaches both intend to pave the way to understanding and formalization, there is little empirical evidence concerning the strengths and weaknesses of the two different strategies. The aim of this study is to explore the feasibility and effectiveness of a learning strategy for understanding through the use of the number line as a tool in the process of formalization. Against this background, the following research questions are central: How do student learning-processes develop in an experimental curriculum in which the number line is used as a tool, and what are the learning outcomes of the experimental program as compared to the program in the control group? As regards the acquisition of number sense, the hypothesis advanced here is that fractions can be taught in meaningful ways through the use of the number line. This predicts that students in the experimental group will outperform their counterparts in the control group, especially in terms of insight and flexibility.

2. Theoretical background

Mental models such as the number line or the circle are important tools for mathematical problem solving and insight. However, according to Perkins & Unger (1999) the identification of insight with the possession of mental representations goes too far. The possession of a model, for example that of the number line, is not sufficient. In order to make good use of models, it is not just possession that is required, but also the ability to apply such models in various problem

situations. Thus at least two important questions are generated: (i) which model is most suitable for the representation of mathematical operations such as multiplication and division of fractions, and (ii) how can mathematics education provide guidance in the best use of these models? Both questions are important for mathematical insight and reasoning. A preliminary question concerns the dilemma of 'providing versus generating'. Which is more effective: to provide students with the correct models, such as the circle or the number line, or to provide students with the opportunity to develop their own models? Mayer's studies (1989) have shown that conceptual models provided by curriculum developers and teachers may serve quite well in providing creative solutions to transfer problems. However, starting from the assumption that self-generated models might be more suited to particular students, there are good reasons to suppose that the fundamental distinction between 'providing or generating' should be further investigated. Such models are not only possessed but also invented; that is, via the process of 'guided reinvention'. It could be argued that learner-constructed models enable students to think more creatively and competently, and that they consequently lead to more insightful behavior. Mayer (1989) touched upon this important issue by referring to research findings that showed that high-aptitude students do not profit from models provided by the teacher, since they come to the lessons either with already existing models, or the ability to rapidly construct such models. For these students the teacher-generated models may conflict with their self-generated, more sophisticated models.

These kinds of general questions also appear in the context of mathematics education, in particular in the teaching and learning of fractions. In past decades formal arithmetic with fractions in primary schools generally resulted in the great majority of students having to follow meaningless rules of calculation. Hart (1981) showed the kinds of difficulties that students encounter: 'fractions are not just an easy step from whole numbers' (p.81). Several years later Hart observed how things might go wrong: 'The results showed overwhelmingly that the recipients of a series of practical experiences leading up to a formalization did not appreciate the fact that the latter was a syntheses of the former.' (Hart, 1987; 410). In other words, there seems to be a considerable gap between practical experiences and formal calculations with fractions. As a consequence Hart recommends shifting the subject of formal reasoning with fractions from primary school to secondary education. A similar consensus within the community of mathematics educators and researchers led to formal reasoning with fractions being moved to secondary education (12+) in the National curriculum standards for primary education in the Netherlands. These standards indicate that learning fractions in primary schools should be restricted to learning the 'language of fractions', to using 'elementary' fractions in simple contexts, and to positioning fractions on a number line (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994).

Thus, in some circles the mathematics education community seems resigned to the fact that there are enormous problems in the teaching of formal mathematics to primary school students. This has led to a tendency for postponing the introduction of more abstract calculations requiring formal reasoning to later stages in secondary education. By contrast, the basic hypothesis advanced in the present study is that fractions can be taught in meaningful ways through the use of the number line as a key-model.

Let's now turn to a more in depth analysis of the literature concerning formalization in learning fractions. Carraher and Schliemann show a specific misconception in interpreting fractions. They argued that many students see the fractions' numerator and denominator as a number of pieces, in part-whole-situations were the fraction is to operate on discrete objects, e.g. the shaded part in figure 1 is seen as 2/12 rather than 1/6. Carraher and Schliemann conclude that the relative character of fractions makes them hard to grasp. Novillis Larson (1980) describes similar problems when 13 to 14 year old students are asked to position fractions on a number line that has length '2' or is divided in more pieces than the numerator suggests.

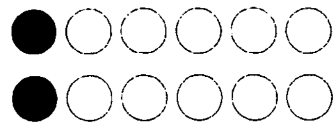

figure 1 : shaded part is seen as 2/12 rather than 1/6

Hasemann gives four arguments why fractions can be considered one of the most difficult subjects in primary school:

'- fractions are used less often in daily life and are less easily described than natural numbers,
- the written form of the fractions is comparatively complicated,
- it is not easy to put the fractions in order size on the number line,
- for the arithmetic of fractions there exist many rules, and these are more complicated than those for natural numbers.' (p. 71)

Hart presented students both situated problems and bare 'sums' on fractions. She shows that children tend not to formalize informal situation-bound solutions. Hart observed:

'There appeared to be no connection in many children's minds between the problem and the 'sum' since they could successfully deal with the problem but could not apply the same method to the computation. It was as if two completely different types of mathematics were involved, one where the children could use common sense, the other where they had to remember a rule.' (p. 67)

Hart thus shows one of the main issues in learning formal operations with fractions. To overcome these problems – where students show a strict separation between formal calculations and solving problems in situations Bezuk & Bieck (1993) plea for exploring informal notions of fractions with children, before making a start with formal operations. They underline the importance of attaching a meaning to fractions

From yet another perspective, realistic mathematics education as it has developed in the Netherlands in the last 30 years, can be seen as a reaction to the learning of meaningless mathematics in education (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996). Starting from what is called the 'realistic' paradigm, Streefland (1990) developed a new curriculum on fractions in the late 80s. With Bezuk and Bieck he emphasized the necessity of confronting students with meaningful situations in order to force them to generating their own fractions and language for fractions.

In the last ten to fifteen years Streefland's fractions curriculum has found its way into most Dutch mathematics primary school textbooks. Streefland proposed fair sharing as the main fraction-generating activity. In this context, sharing three pizzas among four children may be regarded as a prototype of this kind of situation. He showed how fair sharing stimulates the development of a language of fractions. In particular, as students compare the different findings, when the results of the sharing process are discussed (figure 2),

- all pizzas are divided into four pieces, with each child receiving three pieces of a quarter of a pizza;
- two pizzas are divided into two and one is divided into four pieces, resulting in one half-pizza and one quarter-pizza per child;
- three of the children give one-quarter pizza to the fourth child;
- etc.

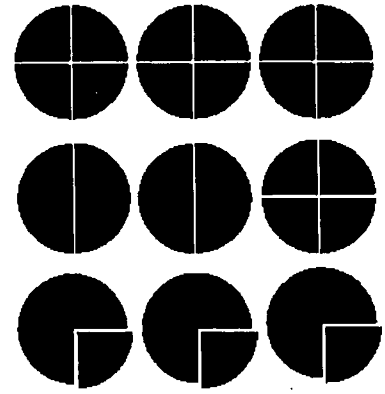

figure 2: results of fair sharing

In the deliberation of the various solutions, the various pieces are named as fractions and simple relations between the fractions are explored. Next, Streefland related fair sharing and ratio. He asked the students to compare situations of fair sharing and grouped equivalent situations in a ratio-table:

pizzas	3	6	9	12	15
children	4	8	12	16	20

In this way equivalent fractions emerge and the road is paved for formal reasoning with fractions. However, fractions as *objects* are not clearly visualized in this way. In the created ratio-situation children have to deal with whole numbers. In other words, by starting with fair sharing, the long road leading from translating fractions in fair-sharing situations into ratio and back is needed to make the formal calculations possible. Moreover, the approach presented does not support the formal multiplication of fractions.[1]

Although Streefland's ideas reflect opinions held only a few years ago, a number of developments would seem to demand further consideration of the way fractions are taught. For example, there is a strong tendency these days to regard the learning of fractions from the point of view of number sense acquisition (Mcintosh, Reys & Reys, 1992). Moreover, National Curriculum standards for primary education in the Netherlands indicate that learning fractions in primary school should include positioning fractions on a number line (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). There is some evidence that using a bar as a model and a number line as abstraction of the bar can be profitably incorporated into a curriculum that aims at number sense (Keijzer, 1997). Furthermore, Freudenthal (1973) considered the number line as 'the most valuable tool which modern ways of teaching arithmetic have borrowed from modern mathematics (...) it can be an excellent means of visualizing the four main arithmetical operations.' (1973, 211). However, the number line as a model for fractions hardly fits in with the approach Streefland suggested. Given the educational level under discussion, situations of fair sharing of circle-objects (pizzas etc.) and ratios are not isomorphic with fractions as points on a number line.

3. The curricula

3.1 Realistic mathematics education

In the experimental setting the curriculum is an adaptation of the 'Fractiongazette' (De Breukenbode) (Buys, Bokhove, Keijzer, Lek, Noteboom & Treffers, 1996). The control setting uses the textbook series from the school where our experiments were conducted, entitled 'The world in numbers' [De wereld in getallen] (Huitema et al., n.d.). Both curricula have been developed in the Dutch tradition of realistic mathematics education (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996).

Consequently, both in the experimental curriculum and in the control group curriculum the focus of attention is on learning in context, in order to generate schemes and models. These, in turn, are used to support the application of mathematical knowledge in other situations, as well as in the development of formal mathematics. Moreover, in realistic mathematics education – and thus in both curricula – mathematics is considered to be a human activity, which is learned in interaction with others and is taught in such a way that various subjects are interwoven, to prevent the development of disconnected, inapplicable knowledge.

3.2 The experimental curriculum

3.2.1 Measuring to develop fraction language

Within the framework of realistic mathematics education, three key features are formulated for the newly developed 'Fractiongazette' curriculum. These are as follows:

- The curriculum is directed towards the acquisition of number sense: students learn to give meaning to fractions in various kinds of situational contexts, develop a good notion of the size of fractions, and learn to handle fractions in simple applications.
- Different situational contexts and models are used: two types of situations, dividing and measuring, lead to the bar and the number line as central models for fractions.
- Students are offered the opportunity to present their approaches at several levels: initially, when confronted with fraction problems, they opt for informal approaches; these are followed by semi-formal and formal solutions, which are imbedded in the informal approaches. Thus the students are challenged to reach approaches at higher levels.

Consequently, as in the case of the curriculum presented by Streefland, attention is focused on the meaning of fractions and the development of a language of fractions. However, Streefland mainly used situations of fair sharing to generate fractions as well as the language of fractions, while in the curriculum developed by Buys et al. (1996) that role is taken by measurement contexts.

In one of the first lessons the students are given a bar, called the 'Amsterdam foot' (abbreviated av). They are subsequently invited to measure objects in the classroom with this new measuring instrument. The 'Amsterdam foot', however, is not suitable for making precise measurements. So the students are invited to fold the bar before taking new measurements.

The folded bars result in informal and semi-formal notations of the measured lengths, showing how the students use their fraction language (figure 3 and figure 4).

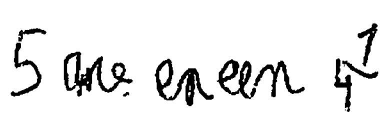

figure 3: Students' work, semi-formal notation. '5 av (Amsterdam foot) and a ¼'

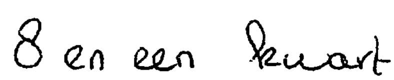

figure 4: Students' work, informal notation. '8 and a quarter'

Discussing these, and similar measurements finally results in a standard notation for fractions. These activities also stimulate further investigation of the relations between fractions. This is the case, for example, when measurement results do not seem to tally, and a result such as 'a half and a quarter' needs to be compared with 'three quarters'. In such typical situations, the students have to give meaning to fractions, and the size of fractions is at the center of the students' attention. Moreover, by the use of the bar in this type of context, an extension of fractions to positions on a number line is at hand. Also, in contexts such as the one described students are free to develop their own language of fractions. However, since the results needs to be communicated frequently, some consensus is

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required.

3.2.2 Comparing fractions

At the next curriculum stage the exploration of the language of fractions provides a good reason for considering relations between fractions, such as '3/5 is greater than ½'. As the number line is developed as a tool for making comparisons, students are free to decide on comparison strategies of their own. However, the fractions to be compared are selected in such a way that informal strategies are somewhat discouraged after some time, while other (more formal) strategies are encouraged. At a certain point in the curriculum students develop the following strategies for comparing fractions:[2]

- comparing fractions at a glance;
- comparing fractions with equal denominators, by counting the pieces;
- comparing fractions with equal numerators, by considering the size of the pieces;
- comparing fractions with ½;
- comparing fractions with 1;
- comparing fractions by using equivalent fractions.

One of the activities at this stage of the curriculum is working with the computer game 'treasure-digging'[3]. Here the students are first offered a fraction and subsequently invited to look for this fraction by clicking on the number line. Every attempt leads to the fraction being shown at the assigned position (figure 5). In this way the students are offered 'anchor points' facilitating the remainder of the searching process.[4]

figure 5: Treasure digging.
The first fraction found is 5/9. This one is greater than 2/5 as it is greater than 1/2. Than the smaller fractions 1/3 and 3/8 are tried. Finally, 2/5 is found.

3.2.3 Equivalent fractions as a field of research for students

At a certain point, strategies for the comparison of fractions lead to the consideration of equivalent fractions. This forms the main topic in the third and final stage of the curriculum. Here, equivalence of fractions becomes a field of research for the students. By introducing the 'Fraction-lift' (figure 6) a start is made with the discussion of formal relations between fractions.[5] The 'Fraction-lift' transports fractions within a building of fractions; numbered lifts take fractions to the place where they live. But since fractions live on many different floors, many lifts are necessary. In this particular context the number of the lift corresponds to the number of stops that the lift makes. The 2-lift stops twice (at ½ and 1), the 3-lift stops three times (at 1/3, 2/3 and 1), the 4-lift stops four times (at ¼, 2/4, ¾ and 1), etc. This means, for example, that the 2-lift can transport ½ to the appropriate position and the 3-lift can move 1/3 and 2/3 to their floors. In addition, the 6-lift can be used to transport both ½ and 1/3.

Next, students start to look for more lifts stopping at a specific floor, for instance by starting to search for lifts that stop at 6/7. Initially, they find the 14-lift and 28-lift by doubling the number 7. Finally, one of the students discovers that every number in the 7-times table can be used.

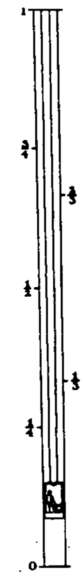

figure 6: the Fraction-lift

In the experimental curriculum students are explicitly allowed to solve problems at several levels. This, in fact, could also be considered as one of the key features of the curriculum. By emphasizing the number line as model of fractions one of the levels that is at hand is making estimations. We therefore introduced a second computer game that allowed students both making estimations or other semi-formal or formal arguments. The game is a memory game. The students are invited to turn over three cards. These cards contain fractions between 0 and 1. We tell the students that it is up to them to pick the cards turned over or not. If they do take the card they will receive three points – one for every card. However, when the students take the cards a little train will start to ride alongside a number line. The length of the ride depends on the difference between the sum of the fractions and 1. If the sum of the fractions is 1 1/6 the train will only move 1/6. However, when the fractions 4/5, ½ and 17/20 are taken (as can be done in the situation displayed in figure 7), the little train will ride to the end of the line as the fractions sum to 2 3/20.
The game is over when the train drops of the number line (at 1). So it might be an idea not to take 4/5, ½ and 17/20; in this situation you better pass. Because in that case the train will only move a distance of 1/10 (figure 7).

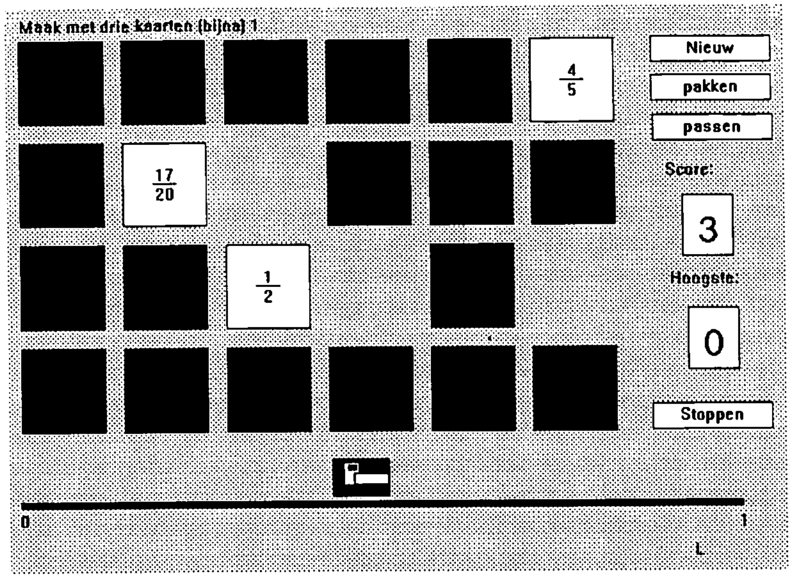

figure 7: Memory game: make (nearly) 1 with three cards
Students can choose between 'pakken' [take] or 'passen' [pass]. The buttons 'Nieuw' [New] and 'Stoppen' [Stop] are used to start a new game or to stop.

We observed how students use several strategies to decide whether the sum of the fractions turned over estimate 1.[6] They estimate the resulting sum 'at a glance', compare fractions with 1 and ½ and use equivalent fractions.

3.3 The control group curriculum

3.3.1 Situational contexts to develop 'fraction language'

Although all curriculum subjects are connected in 'The world in numbers', the part of the curriculum that is aimed at fractions can be separated relatively easily. Thus isolated, the fractions curriculum can be considered in two stages. In the first stage we see part-whole situations, situations of fair sharing and situations in which the fraction operates on a quantity. These situations are used to learn and explore the language of fractions. At first simple unit-fractions are considered. The students are meant to make divisions in an intuitive manner. The different solutions that emerge form a basis for discussion of the various descriptions in fractions.

3.3.2 Exploring equivalent fractions

In the next curriculum stage students explore the equivalence of fractions. The situations used here include using (double indexed) bars to compare fractions (figure 8) and exploring divisions in chocolate bars (figure 9). Thus students are challenged to find a proper quantity to allow two or more fractions to operate upon. This enables them, among other things, to compare fractions. For example, since 12 can be divided by both 3 and 4, both 2/3 and ¾ can be taken from 12. Given that 2/3 of 12 is 8 and ¾ of 12 is 9, it may be concluded that ¾ is greater.[7]

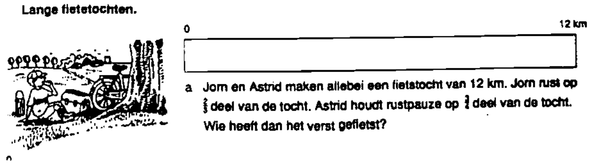

figure 8: Long bicycle tours
Jorn and Astrid both make a tour of 12 km. Jorn takes a rest at 2/3 of the tour. Astrid takes her break at ¾ of the tour. Who made the longest ride?

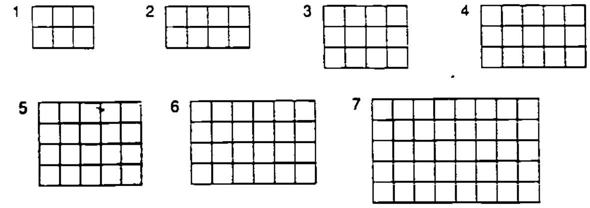

figure 9: Choose your chocolate bar so that you can take ½ and 1/3.

Fair sharing is also used to explore formal relations between fractions, for example by considering the results of fair sharing in different ways.

When we observed students working on these problems in the control group, we never noticed any real interaction between students and between student and teacher. The teacher helped the students individually, and in doing so touched on formal relations, but never allowed them to reflect more deeply on these relations. Many researchers have emphasized the importance of reflecting on approaches and solutions to reach higher-level knowledge (Wittmann, 1981; Van Hiele, 1986; Nesher, 1986; Von Glasersfeld, 1987; Treffers, 1987; Lawler, 1990; Dubinsky,1991; Freudenthal, 1991; Herfs, Mertens, Perrenet & Terwel,1991; Terwel, Herfs, Mertens & Perrenet, 1994; Terwel, 1994; Forman & Fyfe, 1998).

3.3.3 Other activities

In the control group two curricula on fractions were used side by side. Next to the 'The world in numbers' curriculum, individual lessons with divided squares and circles were used to learn fraction language in part-whole situations, and to explore equivalence of fractions. This generally resulted in low-level solutions, as far as equivalent fractions were (or should be) concerned, since students usually got stuck in inching and pinching with concrete materials.

3.4 A comparison of both curricula

When we compare the teaching of fractions in the experimental curriculum with the control group curriculum, considerable differences emerge. These differences concern both the design and the contents of the curricula. As regards the design of the curricula, we found that the control group students in general work individually. Interaction takes place almost exclusively with the teacher, hardly ever with fellow students. When the students work in grouped table sets, they are not engaged in co-operative learning, but are merely working on a similar task. The educational setting in the experimental group, on the other hand, may be characterized as containing whole-class discussions, since there is interaction both between students and between teacher and students. Moreover, the tasks are designed in such a way that students are encouraged to look for meanings and (formal) relations between fractions. In the control group there is little explicit attention to level raising (cf. Dekker, 1991).

Comparing the content of the two curricula, we observe the central role of the number line in the experimental curriculum, whereas in the control group curriculum (divided) bars and circles are means to gain access to fractions. We also see that the number line is used in the experimental group to explore strategies for comparing fractions. Moreover, these strategies converge on the use of equivalent fractions for those students who are capable of understanding the formal mathematics involved, and in situations in which more informal approaches are inappropriate. By contrast, in the control group operator situations and manipulations with pre-divided circles are explored for the investigation of equivalent fractions.

4. Research questions and hypothesis

Although both curricula discussed here cover both stages in the learning of fractions, in which students get acquainted with fractions, as well as stages for which formal reasoning with fractions is the objective, we initially tried to restrict ourselves to the latter, i.e., formal reasoning with fractions. In this research we look primarily at the two curricula for formal reasoning with fractions. This starting point, however, forces us to consider all stages in the learning process. Bergeron, Herscovics and Bergeron (1987) describe the complexity of connections between different elements in a learning process towards abstract mathematics as follows:

'The model of understanding we have used as our theoretical framework suggests four levels of comprehension (intuitive understanding, procedural understanding, abstraction and formalization). However, it would be a mistake to perceive it as a linear model in the sense that a given level can only be achieved after all the steps of the preceding level have been covered. (...) The child evolves simultaneously at many levels.' (p. 359)

Mason (1989) also considers the relation between different levels of reasoning. He regards abstraction, i.e. decontextualizing[8], as a shift of attention.

'Once abstraction is recognized as a shift of attention, it is possible to be of direct assistance to students, by constructing activities which call upon their powers of generalizing to express in their own terms what is the same about a number of, for them, situations, for the author, examples, and then explicitly draw attention to the movement in which the expressions become manipulable entities.' (p. 7)

If we are to take these points of view seriously, we cannot limit our research to only one stage of the curricula we described earlier. We must therefore consider both curricula in their entirety. Against this background we formulated the following research problem:
How do learning-processes of the students who are taught in the experimental fractions course progress and what are the effects of the experimental course compared to the control group course?

Both curricula aim at the development of formal reasoning with fractions. By contrast, national curriculum standards in the Netherlands emphasize that this type of formal reasoning should not be an objective for all primary school students. However, many students in primary school grades 7 and 8 (10-12 year) seem to be capable, at a suitable level, of understanding the formal mathematics involved in such formal reasoning. We consequently formulated the following two research hypotheses:

Hypothesis 1: General mathematical proficiency hypothesis
Learning-processes in the experimental group progress such that a greater growth in general mathematical skills can be demonstrated, when this growth is compared to that of students in the control group, especially for those students who learn mathematics relatively easily.

Hypothesis 2: Proficiency in fractions hypothesis
Learning-processes in the experimental group progress such a way that a greater growth in 'fraction numeracy' can be demonstrated, when this growth is compared to that of students in the control group, especially for those students who learn mathematics relatively easily.

5. Methods

Freudenthal (1991) and Gravemeijer (1994) consider the development of mathematics education as a cyclic process, starting with the developer's thought-experiments, followed by teaching-experiments with students. Reflections on the teaching-experiments lead to reconsideration by the developer and an improved version of the curriculum developed. In a sense, the present research describes only one of the stages in this cyclic process. A newly developed curriculum on fractions is considered and discussed here in a quasi-experimental

setting (Cook & Campbell, 1979). An experimental group followed the newly developed curriculum and a control group followed a curriculum from the textbook series 'The world in numbers' (Huitema, et al., n.d.). Both groups were followed for the duration of one school year. The analysis of the work by the experimental group students and the control group leads to certain conclusions about both curricula, which could form the basis for further research.

Thus this research can be considered as a quasi-experimental design and can be typified as non-equivalent control group design with pre-test and post-test (Cook and Campbell, 1979) and can be schematized as follows (p. 124):

table 1 : Basic design of the study

O_1	Xe	O_2
O_1	Xc	O_2

The first row shows the development of the experimental group and the second row of the table that of the control group. The pre-test and post-test are equal for both groups. In this experimental setting this scheme is somewhat extended in two steps. In the first step we introduce an extra test halfway the experimental year:

table 2: Specified design of the study

O_1	Xe_1	O_2	Xe_2	O_3
O_1	Xc_1	O_2	Xc_2	O_3

In this setting O_1, O_2 and O_3 are standardized tests that are used to test the general mathematical skills of the students[9]. As these test are interrelated by scaling-techniques the growth in mathematical skills of students in both experimental group and control group can be quantified over the year the experiment takes place. After the pre-test a matching-process resulted in equivalent experimental and control group. The experimental group undergoes the experimental program, Xe. The control group went through the original program on fractions in the school, Xc.

Standardized individual interviews provide a more qualitative measure of the growth of the students in what could be described as 'fraction numeracy' (cf. Paulos, 1988; Mcintosh, Reys and Reys, 1992; Bokhove, Buys (ed.), Keijzer, Lek, Noteboom and Treffers, 1996). During the experiment every student in the control group and the experimental group is interviewed three times. The topics of the interviews are inspired on the national curriculum standards on fractions. Moreover these subjects play a significant role in both experimental curriculum, which developed as an extension to the course 'the Fractiongazette' (De Breukenbode) (Bokhove, Buys (ed.), Keijzer, Lek, Noteboom and Treffers, 1996) and the curriculum in the control group, which is the school-program (mainly) based on 'The world in numbers' (De wereld in getallen) (Huitema et al., n.d.). The first interview was developed to provide information on the 'fraction language' of the students and the second interview in a similar way dealt with comparing fractions. The students were offered a number of problems and were invited to solve these aloud. Moreover they were asked to explain their approaches, in case these were not clear, and received some standardized help, when this seemed necessary. This kind of help was intended to explore the Zone of Proximal Development (ZPD) (Hedegaard, 1990; Van Oers, 1996; Tharp & Gallimore, 1998). The third interview was directed at applying knowledge on fractions. Here students first had to do the problems in a written test[10]. In the interview that followed students were invited to explain their approaches. Here too the students received some standardized help, in case this seemed necessary in exploring the ZPD.

When we add the interviews (I_1, I_2 and I_3) to the schematized design of the research, we obtain the following sequence of research activities in both experimental group (first row) and control group (second row):

table 3: Final specification of the design of the study

O_1	Xe_1	I_1	Xe_2	O_2	Xe_3	I_2	Xe_4	O_3	I_3
O_1	Xc_1	I_1	Xc_2	O_2	Xc_3	I_2	Xc_4	O_3	I_3

All students involved in this research were followed extensively and thoroughly in their learning of fractions. In this respect the research can be typified as a *multiple case study* (Yin, 1984). Yin considers a case-study-design appropriate here:

'The most important (reason to perform a case-study) is to *explain* the causal links in real-life interventions that are too complex for the survey or experimental strategies.' (p. 25).

Consequently in this research we also consider the case-study-design. This adds observations made during the learning process to possible explaining factors in the study. While standardized tests and interviews provide both quantitative and qualitative data, these observations are merely a source of qualitative information. Moreover tests that provide data on general mathematical skills should be distinguished more clearly from curriculum specific skills. These considerations on quantitative and qualitative aspects of the research lead to the need for an alternative way to schematize the project. For this reason we suggest the following design to schematize the research[11]:

table 4: Analysis of qualitative and quantitative data

	quantitative data	qualitative data
general mathematical skills	standardized test: O_1, O_2 and O_3	standardized test: O_1, O_2 and O_3
curriculum specific skills	standardized interviews: I_1, I_2 and I_3	standardized interviews: I_1, I_2 and I_3 observations during the learning process

In the next paragraph we will utilize this design to analyze the data from standardized tests for general mathematical skills and standardized interviews for curriculum specific skills. In this analyses quantitative and qualitative data are interrelated. Moreover the observations that are made during the learning process complete the picture, as they provide possible explanations for the observed effects of the experimental curriculum compared to the effects of curriculum used in the control group[12].

6. Data and analysis

6.1 Matching students

The experimental and control groups are constructed in such a way that they are both composed of grade 6 students (9 to 10 years) from two parallel groups, taken from a school in the north of Amsterdam. Since the two groups involved in the experiments are mixed groups, with students from grades 6, 7 and 8 in both groups, there are only twelve students from grade 6 in each group. Only ten of these students participated in the pre-test. The matching procedure now consisted of the following steps[13]:

- a pre-test to establish general mathematical skills of the students;
- making an initial one-to-one matching of students;
- interviews with the teachers of the students to retrieve general characteristics of the students (background, conduct, skills in mathematics and command of language[14]);

- utilizing this information to finally match the students from both groups one-to-one;
- random assignment of one of the groups as experimental group and the other as control group.

Analysis of the test-results facilitated the initial matching, while the interviews with the teachers resulted in additional arguments for the initial matching. One of the groups was assigned as experimental group and the other as control group. As mentioned earlier, the experimental group followed the experimental program, while the control group followed the curriculum based on the teachers' interpretation of the textbook series used in the school.

6.2 General mathematical skills

During this one-year period the students were pre-tested, and two subsequent post-tests on general mathematical skills were administered. In addition, a series of three interviews on curriculum-specific topics were held with every student. The general mathematics tests consisted of a sub-test on 'numbers and operations' and on 'measuring and geometry'. We compared the group scores, using a *paired-samples t-test*. This test was appropriate, as the observations for each matched pair were made under the same conditions[15]. We took the difference in skill as a measure of the difference in development of the matched students and also took the group-scores of the matched couples into account. Doing so, we could not establish significant differences between the experimental group and the control group. If we, however, restrict the paired t-test to those students that are normal performing in mathematics[16], we do find significant differences[17]. We therefore decided to compare the two groups in this study using a regression analyses. In figure 10 we plotted the individual results of the pre-test on 'numbers and operations' against those of the post-test in the same subject. Moreover for these tests we calculated the regression-lines and the prediction intervals for single observations (confidence level = 95 %). Similarly the chart in figure 11 displays the development of the students in 'measuring and geometry'.

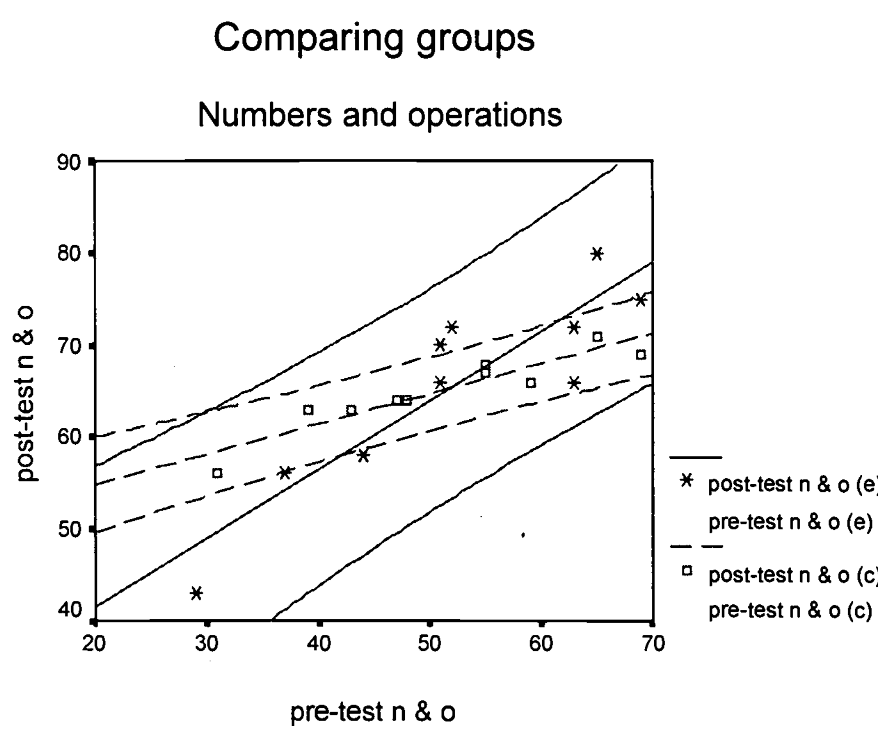

figure 10: Development in skills in 'numbers and operations' in experimental group (N = 10) and control group (N = 10). The fit method used is linear regression, with prediction intervals for single observations (confidence level = 95 %) for both control group (c) and experimental group (e). Regression equations: Experimental condition: post-test = .751 pre-test + 26.431;

Control condition: post-test = .329 pre-test + 48.288

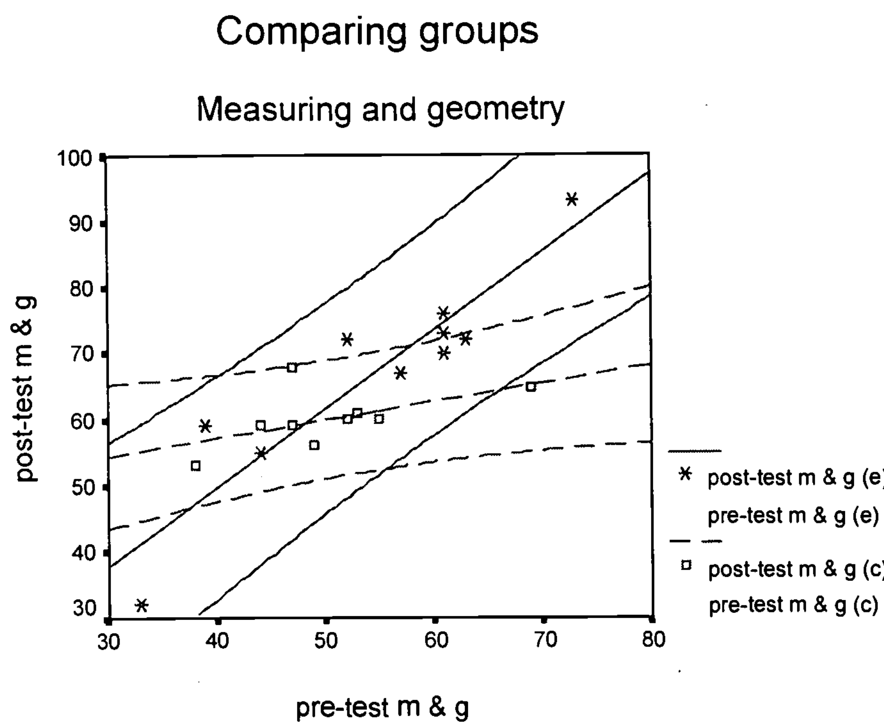

figure 11: Development in skills in 'measuring and geometry' in experimental group (N = 10) and control group (N = 10). The fit method used is linear regression, with prediction intervals for single observations (confidence level = 95 %) for both control group (c) and experimental group (e). Regression equations: Experimental condition: post-test = 1.193 pre-test + 1.989; Control condition: post-test = .278 pre-test + 46.043

The charts in figure 10 and figure 11 suggest differential effects in the development of general mathematical skills in the experimental group as compared to the control group. The better performing students in the experimental group in the pre-test outperform their matched counterparts in the control group. In addition the graphs suggest that the less well performing students in the experimental group achieve a little less than their fellow-students in the control group. [18]

We hypothesized that the learning-processes of the students in the experimental group should progress in such a way that a greater increase in general mathematical skills could be demonstrated, when this growth is related to that of the students in the control group. When we consider the whole groups, we established that although the test-results are somewhat better for the students in the experimental group, the test-results show no significant difference between the (development of the) general mathematical skills of the students in the experimental group as compared to these skills of students in the control group. [19]
We also hypothesized that especially better achieving students in mathematics should perform better on general mathematical tasks. We showed a tendency towards differential effects in comparing the pre-test and post-test scores in a regression analysis of the students in the two conditions. Although no significant results could be established, this analysis of the data suggests that those students in experimental group that acquire mathematics with relative ease, achieved better results on the post-tests, compared to their matched counterparts in the control group. In other words, we observed a so-called 'Matthew-effect' for the experimental condition (Hoek, Terwel & Van der Eeden, 1997; Hoek, 1998). We thus observe some indications that partial confirm of our hypotheses on general mathematical skills.

6.3 Proficiency in fractions

6.3.1 Quantitative data

Only one of the general mathematical ability tests, the post-test on 'numbers and operations', contained a few items (10 % of the test) on fractions. In the other tests no items on fractions could be found. As therefore these test do not show the progress in the students' proficiency in fractions, three interviews[20] were developed to recover the students' skills in fractions. The first interview is held after three and a half months. In appendix 1 we show one of the three problems from the first interview (with the standardized help given by the interviewer and an analysis of possible reactions of the students). The problems in the first interview demand a reasonable knowledge of the 'language of fractions'. However, as indicated in appendix 1, the problems can be solved at different levels, thus facilitating most students to yield admissible solutions. Also in appendix 1 we displayed two the four problems from the second interview, that is held three months later. In this interview comparing fractions was the subject. Moreover by presenting the students rather open problems we facilitated them to show their mathematical attitude, for instance by manifesting flexibility in using knowledge on fractions. On the other hand these open problems made it possible for the students to come forward with solutions that suits their level and preferences. We show how standardized help is offered here. In appendix 1 we displayed also two of the seven problems from the third interview, again held three months later. Here the students could show their capabilities in applying their knowledge on fractions. In this case the students first did the problems in a written test. The solutions from this written test formed the starting point for the interviews. As shown in appendix 1 the students were asked to explain their approaches and here too we offered standardized help, if the students did not come up with a correct solution within reasonable time.

This research setting offers us the opportunity to compare the development of the students in the matched couples. Moreover we are able to show the influence of the help offered. In other words, we here show how both curricula facilitated the students in solving the problems and to what extent they were prepared to use the help that was offered. This help in general was constructed to aid the students to utilize and enlarge their flexibility in understanding fractions. Consequently in the interviews the students are encouraged to review their approaches and relate the situations to more familiar ones. As a consequence mathematics comes forward here as a subject to discuss about. Therefore especially the responses of the students to the offered help could be considered as an indicator of acquired number sense on fractions.[21] As mentioned before, this kind of help was intended to explore the Zone of Proximal Development (ZPD) (Van Oers, 1996).

As the third interview is the interview held after one year, one can consider this interview as indicator of the differences in skills in fractions between students in the two groups. The chart in figure 12 suggests how the students in the experimental group outperform their fellow students in the control group; their number of correct solutions in the interview is about one more than their matched counterparts in the control group. Therefore the graphs provides a first indication that the skills of the students in applying fractions in non-typical situations in the two groups differ.

Comparing groups

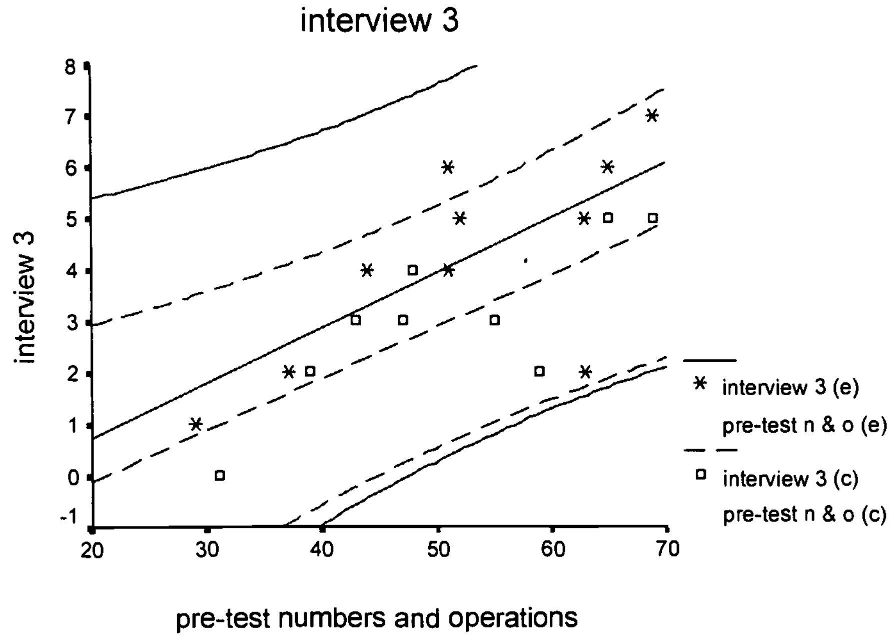

figure 12: Results of the third interview in experimental group (N = 10) and control group (N = 10). The number of correct answers in the interview (with help) is plotted against the results of the students on the pre-test in 'numbers and operations'. The fit method used is linear regression, with prediction intervals for single observations (confidence level = 95 %) for both control group (c) and experimental group (e). Regression equations: Experimental condition: interview 3 = .107 pre-test – 1.398; Control condition: interview 3 = .100 pre-test – 2.115

We compare the quantitative data from the interviews more thoroughly. We therefore consider in each interview the number of correct answers given with and without standardized help. Here again we use a *paired sample t-test* to compare the results. From table 5 we learn that in all but one case the students in the experimental group perform better than their counterparts in the control group. Especially we see that students in the experimental group in all cases achieve significantly better (sig. level 0.05 or better) when they receive standardized help.

table 5: paired sample t-test, comparing the results on three standardized interviews, where student received no help or standardized help (* significant results at 0.05 level)

pairs	Paired Differences			95% Confidence Interval of the Difference (one tailed)	t	df	Sig.
	Mean	Std. Deviation	Std. Error Mean				
interview 1 (control) – interview 1 (exp)	–1.40	.84	.27	< –.91	–5.250	9	.001*
interview 1. no help (control) – interview 1. no help (exp)	–1.10	.88	.28	< –.59	–3.973	9	.003*
interview 2 (control) – interview 2 (exp)	–1.20	1.40	.44	< –.39	–2.714	9	.024*
interview 2. no help (control) – interview 2. no help (exp)	–1.10	1.10	.35	< –.46	–3.161	9	.012*
interview 3 (control) – interview 3 (exp)	–1.20	1.03	.33	< –.60	–3.674	9	.005*

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Interview 3. no help (control) – Interview 3. no help (exp)	–.50	1.43	.45	< .33	–1.103	9	.299

We hypothesized that the learning-processes of students in the experimental group progress in such a way that a greater increase in 'fraction numeracy' can be demonstrated; when this growth is compared to that of students in the control group. We can clearly state that the problems in the interviews facilitate the students to show their 'fraction numeracy', as they do not aim at standard procedures to solve problems. Moreover, showing number sense (in general) includes reflecting upon given solutions (Mcintosh, Reys and Reys, 1992). Because in the help offered the students in general are invited to consider their approach, we could consider the results from the interviews where students were offered help, as means to operationalize 'fraction numeracy'. If we do so, results we presented in table 5 and figure 12 show that students in the experimental group acquired a greater increase in 'fraction numeracy', confirming our second hypotheses.

6.3.2 Qualitative analyses of students reactions

As 'number sense in fractions' is of a qualitative nature, in the following we present some of the reactions of the students in the three interviews. In our analysis we will compare matched couples of students in their attempt to solve the same problem in one of the interviews (see appendix 1). Next we will compare the 'number sense in fractions' in experimental group and control group in general.

As our previous quantitative analyses show, there is a good reason to distinguish students that are weak in mathematics from those performing without special problems. In doing so we will first analyze the work of Alice (experimental group) and Oliver (control group). These students belong to the 25 % weakest in mathematics compared to a national reference. In the first interview they both struggle with problem 2 (see appendix 1), where a part of a chocolate bar is left over. The students are told that Irene ate 3/5 of the bar. They are asked to reconstruct the whole bar. Both students end up considering the part that is left over as 1/5 and constructing 4/5 to complete the bar (figure 13 and figure 14). When asked to check their answer both students realize that they solved another problem than the one posed.

figure 13: Alice (experimental group): 'I made that see ate 4/5.'

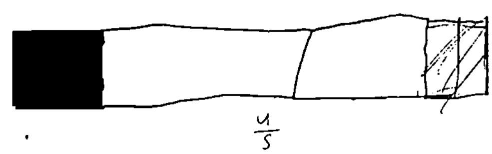

figure 14: Oliver (control group)

In the second interview both Alice and Oliver show having difficulties with problem 1 (appendix 1). They both try to use formal relations between fractions, but do not succeed in doing so. Alice reasons, while comparing the fractions 2/3 and 5/6: 'If you add this, 2/3 and 2/3, you get 4/6. And this (2/3) is 3/12, so it is always more.' The interviewer asks her to explain: 'How do you mean "add"?' The student explains: 'I mean double 2 and also 3.' And again: '5/6 is still more.'

According to Oliver Janneke, who came to 5/6 of the route, came farther than Mieke, who came to 2/3. He explains: 'You have to double both '3' and '2'. '3' is '6' and '2' is '4' and there it says '5'.' The interviewer continues with the distant covered by Jelle, who did a greater distant then Janneke. Oliver reasons: 'This is 5/6, so 10/12.' Although he doubts this answer, he thinks that 10/12 is farther than 5/6.

One of the problems in the third interview is problem 3 (see appendix 1). Here the students are confronted with a part-whole situation. About ¾ part of the students of the Plerikschool

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walk to school. Of the rest half is brought and half comes by bicycle. Alice and Oliver are asked to determine what part of the students comes by bicycle?
Alice first comments on the problem: 'I found this problem odd, since ½ and ½ is already a whole.' The interviewer explains: 'It says "half of the rest".' Then she changes her approach: 'It is "half of a quart", of course!' And, when asked what this is, she answers immediately: 'one eighth.'
Oliver here falls into meaningless formal manipulations with fractions. He states: 'That is 7/6 and the half thereof...' He gets confused by his own approach. The interviewer asks him if he can make a sketch to show what he means. Oliver does not make a real start with this and suddenly comes forward with: 'It's 2/16. And the half thereof. That is logical.' The interviewer asks him to explain. However Oliver cannot tell how he found the fraction 2/16.
Then the interviewer decides to turn over to the standardized help: '¾ of the students walk to school. What part does not walk to school?' Oliver: '7/6.' The interviewer repeats the question and asks Oliver to think the situation over one more time. However, he holds on to his answer 7/6.

If we compare the 'fraction numeracy' of both Alice (experimental group) and Oliver (control group), we see that both students have considerable difficulties in using formal relations between fractions. In the third interview, at the end of the experimental year, Alice shows how she turns to an informal approach, namely using a known numberfact. Oliver continues to use uncomprehended formal calculations with fractions. Moreover, Oliver is not capable in explaining his formal approaches.

Now let us have a closer look at two students, Roxanne and Gina, that perform at an average level (compared to a national reference). These students also form a matched couple. Roxanne is one of the students in the experimental group. In the first interview she tries to solve the second problem ('Irene ate 3/5 of her chocolate bar', see appendix 1). If the interviewer asks her if Irene ate more that half the bar, she replies immediately: 'There are three pieces out of five; two-and-a-half pieces would be half a bar.' If asked what is left, Roxanne tells this is 2/5 of the chocolate bar. She is however not really sure about her answer and therefore decides to make a drawing. Roxanne halves the piece of the chocolate bar on the work sheet by pointing at it and paces this three times behind the drawn chocolate bar with her fingers. She explains: 'Now there are five pieces as there should be.'
In her interview the interviewer also asks Gina, from the control group, if Irene ate more than halve a chocolate bar. She thinks Irene did so. Gina tries to explain: 'Because she ate five third... Oh no, she ate three fifth.' The interviewer responses: 'Why is three fifth more than a halve?' Gina now specifies her answer: 'Because two fifths is a halve.' However, Gina seems to be unsure about this: 'Oh, I don't know...'
When asked what fraction goes with the piece of the chocolate bar that is left, Gina replies: 'One fifth.' When next the interviewer asks her to complete the drawing of the bar, Gina does so by sketching a segment to the right of the piece of the bar on the work sheet. This drawn part of the bar is about twice the length of the piece already there. Gina divides the segment in two and adds another part. When asked to check her answer she establishes that she made ¾. Moreover she cannot think how she should make three fifths. Gina decides to write ¾ under her work (figure 15).

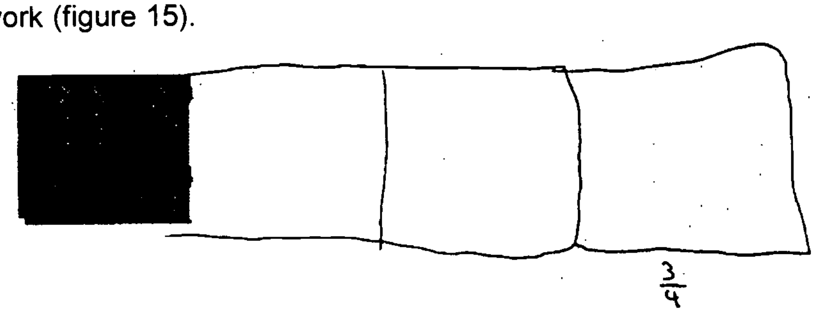

figure 15: Gina's drawing

In the second interview both Roxanne and Gina in their interviews work on the fourth problem ('Find a fractions very close to ¾', see appendix 1). Roxanne, experimental group, immediately suggest 5/8 is a candidate. In her explanation she uses equivalent fractions: 'Twice 3 is 6 and twice 4 is 8. It may not be the same, so I take one before.' The interviewer asks Roxanne if she can find a fraction that is still closer. After a long thought Roxanne mentions the fraction 2/3, but explains that this is just a gamble and that she thinks 5/8 is closer by.
Gina in her interview suggests that 6/8 is a possibility. However, when the interviewer tells her that this one is exactly ¾, Gina tries the fraction 2/4. She explains: 'Because it is one less.' The interviewer asks Gina if she knows of a fraction still closer by ¾. Then, after a short while Gina, with some hesitation in her voice, suggests that 5/8 might be a candidate. When the interviewer asks her to if this one is closer by, Gina explains her approach: 'Because 4/8 is a halve and 2/4 is too.'

The third interview is aimed at applying knowledge on fractions. One of the problems Roxanne and Gina work on in this interview is problem 6 ('At the beginning of her holidays Janita has ƒ 48,- (48 Dutch guilders); 1/6 part thereof she spends on ice-cream and lemonade, 1/4 part she spends on picture postcards and the rest she spends on presents', see appendix 1).
Roxanne, experimental group, when doing the problem in a written test, wrote that Janita spends ¾ of her money on presents. In her interview she explains why this answer cannot be correct: '1/6 of the money is 8 guilders and ¼ is ...'
If she gets stuck here the interviewer asks her to make a drawing. When she does so 1/6 and ¼ end up over each other (figure 16).

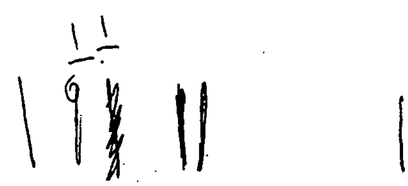

figure 16: Roxanne adding 1/4 and 1/6

Roxanne interprets her sketch: 'Together it is about halve.' When next the interviewer asks her if she can tell a little more about it, Roxanne shows in her drawing that the sum of ¼ and 1/6 is a little less than halve. She concludes: 'Janita has left a little more than halve over to spend on presents.'
Gina did not finish the sixth problem in the written test. Two days later, in her interview she tells the interviewer about the problem: 'I don't get this!' The interviewer next asks her to make a sketch of the situation and to calculate how much money Janita spent on ice cream and lemonade. However, Gina cannot do anything with this help.

If we compare the two students in this matched couple, we see that initially the fraction-language of Roxanne is more developed than that of Gina. While Roxanne easily found the complement of 3/5 and used this information to draw the chocolate bar in the bar-problem in the first interview, Gina confused 2/5 with ½ and lateron with 1/5 (or maybe better: with one piece of any fraction). In the second interview, two months later, both students use formal relations between fractions to get close to ¾. However, in contrast with Roxanne, Gina hesitates a little to do so. Moreover, Roxanne also compares by the look of the fractions.
In the third interview we see how this approach of Roxanne of using several levels at the same time, results in a reasonable solution of adding ¼ and 1/6; it is nearly ½. Gina here probably encounters that she cannot use her formal arithmetic with the fractions involved and drops out. We see how Roxanne, in contrast with Gina, developed a repertoire to tackle the problems on an appropriate level. We consider this as sign of acquired 'fraction

numeracy'.

In the third matched couple we consider here, we find two better performing students[22]. Ines is one of the students in the experimental group and Raven is her counterpart in the control group. In her first interview Ines solves the second problem ('Irene ate 3/5 of her chocolate bar', see appendix 1) with remarkable ease. Before the interviewer can make a start with telling about the situation, Ines answers the (implicit) question: 'Two fifth is over.' Next the interviewer asks if Irene ate more than halve the bar. Ines again replies immediately: 'Yes, because three fifths is more than two fifths.'
Raven also works on this problem in her first interview. When the interviewer asks her if Irene ate more than halve the bar, she hesitates. Initially she tells Irene did eat more than halve a bar, but changes this immediately and finally reaches no answer at all. Moreover, in her reasoning to compare 3/5 with ½ Raven tells that ¾ equals 3/5.
Next the interviewer asks Raven what part of the bar is left and suggests her to make a sketch of the situation. Raven concludes that ¼ is left and draws three pieces next to the drawing on the work sheet. The interviewer asks her to explain. Raven: 'One quarter is over and three quarters are eaten.' The interviewer shows some confusion: the problem stated that three fifth is eaten. This, however, is no problem for Raven, since three quarter and three fifth are the same for her.

In the second interview, Ines and Raven both are confronted with the fourth problem, 'find a fractions very close to ¾' (see appendix 1). Ines, experimental group, when seeing the problem, responds at once: '6/8.' When the interviewer tells her that fractions that equal ¾ are not meant here, Ines tries 4/5 and 2/3. The interviewer asks her if there are any fractions closer by ¾. This elicits Ines to use the equivalent fraction found earlier; she finds 5/8 and 7/8 and explains: 'These are equally close to ¾.' The interviewer asks Ines if these fractions are closer by than 4/5. Ines thinks this is so, she however makes a sketch to be sure (figure 17). Although the drawing is not very successful, Ines uses it to come to a conclusion: 7/8 is closer by.

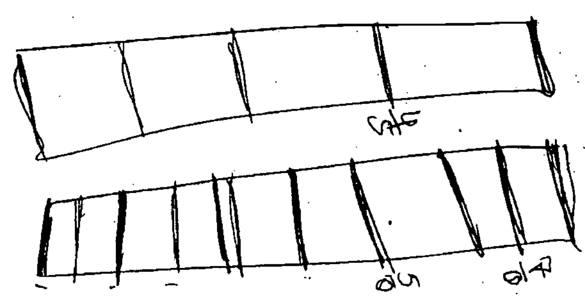

figure 17: Ines compares 5/8, 7/8 and 4/5

Raven, control group, also tries to tackle this problem. She experiences difficulties in doing so. Initially Raven writes 2/4 down as candidate, but she cannot really explain this choice: 'You cannot know where the ¾ is taken from.' The interviewer tries to support her in this operator approach[23]: 'Indeed, you don't know, but you may think of something.' However, Raven is not helped with this remark.
Next the interviewer asks her if she can find another fraction closer by ¾. Raven takes some time to think this over and finally mentions the fraction 5/4. When she tries to explain why this one is close by, Raven again tries her operator approach; once more with little success. And, when asked to choose between 2/4 and 5/4, she takes the first fraction to be closest to ¾.

In their third interview both Ines and Raven are confronted with the sixth problem ('At the beginning of her holidays Janita has ƒ 48,- (48 Dutch guilders); 1/6 part thereof she spends on ice-cream and lemonade, 1/4 part she spends on picture postcards and the rest she spends on presents', see appendix 1). They both did the problem first in a written test. Ines,

experimental group, here answered '7/12'. In her interview she explains how she solved the problem by using equivalent fractions: 'Four times six is twenty-four and the halve of twenty-four is twelve.' And, pointing at the fractions 1/6 and ¼ on the work sheet; 'This is 4/24 and that 6/24, that makes 10/24 or 5/12.' Ines compares this with her answer on the work sheet and thinks she made a mistake. Then she re-reads the problem and concludes that her initial answer was correct.
Raven, Ines's counterpart in the control group, answered '½ part' in her written test. The interviewer asks her how she found this answer. Raven: 'First 1/6 of 48... that is 8. And ½ thereof... I don't get it.' The interviewer asks her to make a sketch of the situation, however, Raven fails in doing this. She cannot solve the problem.

If we compare the observations of Ines, experimental group, and Raven, control group, we see that initially the knowledge of the fraction-language of Ines is more developed than Raven's. In the first interview Ines easily relates 2/5, 3/5 and ½. Raven here reasons only with the number of pieces in the fraction, resulting in equaling ¾ and 3/5. Two months later the students are asked to look for fractions near ¾. Ines shows here how she tries to combine reasoning with equivalent fractions and making a sketch of the situation, when formal reasoning fails. Raven, on the other hand, understands fractions to little to really find a fraction near ¾. She tries to see the fractions as operators, but fails to accomplish this. Finally in the third interview Ines shows she is able to formally add the fractions ¼ and 1/6. Raven, on the other hand, seemingly cannot solve the problem here on any level. Here again, we see how the student in the experimental group Ines developed a repertoire to tackle the problems on an appropriate level. In contrast, the student in the control group, Raven, gets stuck in handling fractions in non-typical situations. The student in the experimental group can handle these situations with relative ease. We again consider this a signal of acquired 'fraction numeracy'.

We hypothesized that the learning-processes of students in the experimental group progress in such a way that a greater increase in 'number sense with fractions' can be demonstrated, when this growth is compared to that of students in the control group. In the previous paragraph we showed that the quantitative data from the interviews largely confirms the hypotheses. In this paragraph we saw that observations made during the interviews provide additional support for our hypotheses.

7. Discussion and Conclusions

7.1 Summary of findings

In this article we compared two curricula on fractions.. Or better: during one year we followed the development of general mathematical skills and proficiency in fractions of students that were taught with the two programs. We observed how one of the programs, the experimental curriculum, made extensive use of the bar and the number line as a model for fractions. Whole class discussions in this program were aimed at students acquiring the language of fractions and next to explore strategies to compare fractions. Reflection on these approaches for some students ended up in using equivalent fractions in those situations where this is appropriate. In the second program, the control curriculum, the principle models for fractions are the bar and the circle. Moreover exploring the operation situation, fair sharing and manipulating pre-divided circles here paved the road to formal reasoning with fractions. In contrast with the experimental group, in the control group students in general worked individual on their tasks. Here interaction is largely limited to teacher-explanations.
We observed that in the group where formal relations between fractions were a subject for class discussion, the students performed significantly better on non-standard tasks on fractions, compared to the group where the students in general worked individual. Moreover we found indications that, if we restrict ourselves to those students that learn mathematics with relative ease, students in the first group (experimental group) after the experimental year outperform their counterparts in the second group (control group) on general mathematical

skills.

7.2 Research design

Developing a new curriculum on fractions was one of the objectives in the research project described here. Therefore here the researcher took three roles:
- he is developer of the experimental curriculum,
- he is the teacher in the group, that follows the experimental curriculum,
- he researches the development of students in both experimental and control group.

Gravemeijer (1994) considers the development of mathematics education as a cyclic process where thought-experiments precede field-experiments. These classroom experiments lead to reflections and another prototype of the curriculum. Gravemeijer thus emphasizes the importance of this combination of roles in developing mathematics education. Yin (1984), however, warns against this participant role of the researcher:

'The major problems related to participant-observation have to do with the potential biases produced. First, the investigator has less ability to work as an external observer and may, at times, have to assume positions or advocacy roles contrary to the interest of good scientific practices. Second, the participant-observer is likely to follow a commonly known phenomenon and become a supporter of the group or organization being studied, if such support did not already exist. Third, the participant role may simply require too much attention relative to the observer role. Thus, the participant-observer may not have sufficient time to take notes or to raise questions about events from different perspectives, as a good observer might.' (p. 87)

We follow Freudenthal (1991) in overcoming Yin's objections against a research design, where the researcher is participant. Freudenthal (with Gravemeijer) here speaks of 'developmental research':

'Developmental research means experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.' (p. 161)

In this paper and elsewhere (Keijzer and Terwel, in preparation) we report on the activities of the researcher. In his teaching and developing he is definitely influencing the results of the research, since this is partially the object of the research! Namely the experimental curriculum that is investigated next to the control curriculum is the result of the developmental activities of the researcher. Moreover we consider the style of teaching of the researcher (as well as the style of teaching of the teacher in the control program) as integral element of the curriculum. This entitles the researcher (in his role as teacher) to take a position and support the developed curriculum, as similarly the teacher in the control group does when teaching in her group.

However, the researcher, when reporting on his findings, should not be biased. This paper therefore shows both quantitative data and qualitative data to support the findings. Moreover this report demonstrates how the researcher, as reporter of research-findings, takes an objective position in displaying the results.

7.3 Problem solving as explaining factor

In a sense learning fractions in the experimental curriculum can be typified as problem solving. Namely students are offered open problem-situations, where various solutions are possible. Moreover group discussion is a means to establish the value of the approaches. Verschaffel (1995) mentions three types of knowledge necessary for problem solving:

1. the flexible use of a rich and well organized, domain-specific knowledge-base;
2. skilled ability to use heuristic methods;
3. metacognition[24].

From the analysis of the two curricula involved in this research we learned that the experimental program offered students opportunities to build a rich knowledge base on fractions and relations between fractions. Moreover problems posed here, challenged students

to explore their knowledge base on fractions. In this manner differences with the control curriculum emerge. Though students in this program too build a knowledge base on fractions, they are not challenged to reflect on relations between fractions.
Several activities in the experimental curriculum aim at using estimating approaches in solving problems with fractions (cf. e.g. Roxanne's approach to add ¼ and 1/6). Furthermore, in class discussions global reasonings are frequently used to test statements on fraction-relations. This provides for a strong heuristic method to handle fractions, which has no equivalent in the control setting.
Verschaffel states that the types of knowledge necessary for problem solving offer major difficulties for those students that are not proficient in mathematics. Perhaps this offers an explanation for the observed differential effects observed concerning the development of general mathematical skills of the students. Namely the students in the experimental group that are proficient in mathematics seize the opportunity to move in a flexible way between fractions and, moreover, know how to transfer this ability to other parts of mathematics, whereas those not proficient in mathematics in the program learn some (usually elementary and low-level) relations between fractions. These low achieving students are not in a position to expand their approaches to other mathematical problems. Students from the control group, on the other hand, are hardly offered the opportunity to acquire the types of knowledge for problem solving, mentioned by Verschaffel, explaining our observation that students that perform well in mathematics have difficulties in solving non-typical problems (both in the interviews on fractions as in the general proficiency tests in mathematics).

Goldin (1998) provides additional arguments to consider problem solving here. He states that internal representations are of major importance:
'We must consider not just *task* structure, but *representational* structure, if we wish to understand how human cognitions change while interacting with such environments.' (p. 146)

He defines five categories of these internal representations:
'- verbal/syntactic systems,
- imagistic systems,
- formal notational systems of mathematics,
- a system of planning, monitoring, and executive control,
- a system of affective representation.' (p. 148)

Goldin argues that all of these are all at stake in problem solving. Moreover he sketches necessary consequences for mathematics education. Exploring mathematics should include building several internal representations in different categories, as...
'The redundancy that this multiple-encoding provides can account not only for the persistence of schemata in long-term memory once they are constructed, but also for the ability of individuals to reconstruct concepts when they seem to have been temporarily forgotten.' (p. 158)

We consider this argument as additional support for the curriculum in the experimental group, since in this program all categories mentioned by Goldin are present. However, in the curriculum in the control group, as we learned from the qualitative data, verbal/syntactic systems are less developed and students lack a well-formed system of monitoring their approaches.

7.4 Problem solving and number sense
One of the main objectives in mathematics education in the Netherlands is to stimulate students to develop 'number-sense' (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). We argued that the program in the experimental group contributed more to the development of number sense than the program in the control group. McIntosh, Reys and Reys (1992) show the relation between acquired number sense and heuristic approaches like checking and discussing one's answer, relating answers to situations they came forth and (more general) reflecting on approaches. From Verschaffel's (1995) point of view we are dealing here with

aspects of problem solving. We showed how the curriculum in the experimental group facilitated students to acquire problem-solving strategies. That is why we interpret the point of view exhibited by McIntosh, Reys and Reys as follows. The experimental program aims at acquiring number sense by students. As a consequence students that perform normally in mathematics obtain skills to solve non-typical problems in mathematics. They therefore outperform their counterparts in the control group in both general mathematical skills and domain specific skills in fractions.

7.5 Developing mathematics education

The research we report on in this paper is intended to make a constructive contribution in developing mathematics education. Stigler and Hiebert (1999) state that research, as we described, is likely not to make any contribution in improving the quality of education, since it is aimed at changing only one element of the whole educational system, while leaving the rest of the system untouched.

However, we showed how changing one single element of mathematics teaching improved the system, as we considered not only the element itself. We regarded learning of fractions from the viewpoint of acquiring number sense. In analyzing reactions of normal performing students we observed effects of the curriculum on general mathematical skills. Moreover we analyzed that making formal mathematics a topic in class discussion could be held responsible for this effect. We therefore agree with Stigler and Hiebert that changing just one element in the educational system will not easily lead to changing the whole system. As a consequence we considered a single element in the mathematics curriculum in perspective of the whole system of mathematics teaching.

7.6 Implications for education and future research

We showed how teaching of formal fractions is influenced by using the number line as central model for fractions and by creating an educational setting, where formal mathematics is discussed in the classroom[25]. Students in this setting developed so-called 'fraction numeracy' and showed better prepared to solve non-typical problems in fractions than students in an educational setting, where formal fractions are mainly based on individually manipulating predivided circles and bars and where there is no interaction between students. Moreover we revealed some transfer of these skills in solving non-typical problems to general mathematical strategies. However, we noticed that, if we consider the development of their general mathematical skills, the more gifted students benefited most from the discussion setting. Though, if we look specifically at proficiency in fractions, we see that all students in this setting perform better than their counterparts that work mainly individually.

We thus showed how taking the number line as main model for fractions facilitated whole class discussions aimed at understanding formal fractions. Students involved in these discussions, especially those that have no particular problems in mathematics, obtain skills in manipulating fractions in a meaningful manner. And, while these students are only about 10 years old, they achieve more than curriculum standards in the Netherlands (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994) indicate they should at the age of 12.

The curriculum standards state that teaching fractions, on the whole, should be restricted to learning the language of fractions and to applying operations with fractions in simple applications. We would like to consider this point of view in its historical perspective. For many years teaching fractions resulted in following meaningless rules of calculation. Moreover fractions are hardly necessary to function in this calculator-era (where most fractions are converted into decimals). History proved that fractions are too difficult for students in primary school and many hours of education wasted on fractions are better used more productively[26], must have been considerations.

As we observed in this study, fractions are indeed difficult for some students (in the educational setting we constructed). The limited skills of these weak performers on general mathematical tasks are weak performers in fractions and moreover seem to be another argument to restrict curriculum standards on fractions to simple applications.

It is here, where this study leaves some questions for future research. These research questions mainly concern the possibilities of students that are less proficient in mathematics. As we noticed, these students seem to benefit less of an educational setting, where formal mathematics is a topic in whole class discussion. If we want all students to benefit from class interaction, when the talk is on formal mathematics, more research is needed to analyze the participation of weak students in these class discussions, to find out what are the possibilities of these weak performers.
The standpoint that the skills of weak performers should lead to fractions in primary school should being limited to fraction language and simple applications, however, does not regard the formation of mathematics as self-reliant subject. It is our opinion that mathematics in primary school should cover both applications in daily life or otherwise recognizable situations and formal mathematics, where the last is a synthesis of the first (Treffers, 1987). In other words: there is actually no place for fractions in the curriculum if they are limited to simple applications only, without specific attention for level raising. We showed how situations of measuring, resulted in placing and comparing fractions on a number line and consequently resulted in formal reasoning with fractions. In our study we found arguments to resist historical and other arguments for the limitation in teaching fractions. We therefore consider the presented results a plea for teaching fractions in a meaningful way, where formal mathematics is a subject for whole class discussion in primary school.

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Notes

[1] Like several others Streefland (1988, 1990) used the rectangle situations for teaching the multiplication of fractions (cf. Freudenthal, 1973; Carpenter, Coburn, Reys and Wilson, 1976). Semadeni (1984) also shows her preference for the rectangle as a model for multiplying fractions. She argues that if multiplying is seen as to determine the area of a rectangle, multiplying fractions arises from the same operation with whole numbers.

[2] Armstrong and Novillis Larson (1995) in their experiment found only two strategies to compare fractions (comparing fractions by counting pieces and by considering the size of the pieces). Further analysis of this experiment however shows that this is presumably due to the tasks Armstrong and Novillis Larson presented the students.
[3] This program was created by Frans van Galen, who earlier created a similar program with whole-number tasks. One of the authors, Ronald Keijzer, proposed to extend the existing program to fractions.
[4] The program only shows fractions that facilitate reasoning on several levels. Therefore only fractions with denominator 2, 3, 4, 5, 6, 8, 9, 10 and 12 get visible.
[5] Initially, the 'Fraction-lift' was Adrian Treffers's idea.
[6] In the memory game perceptions of addition from whole numbers can be extended, when making rough estimations. Namely addition with fractions is thus seen as placing objects in a row, with the length of the row acting as the sum. Moreover the game stimulates the students to relate rough estimations with reasonings to come to more precise answers. This activity therefore could be considered as exploring the 'working of the operation addition'. According to Mcintosh, Reys and Reys (1992) this is an essential element of acquiring number sense.
[7] This approach of finding the right quantity to let fractions operate upon, makes that both subtracting fractions and adding is now at hand. E.g. the result of ¾ - 2/3 can be obtained by reasoning in pieces. If both are taken from a bar with 12 pieces, we actually have to calculate 9 – 8 = 1. That is: the difference is only 1 piece from the 12, which is 1/12.
[8] Van Oers (1998) suggests to use the term 'recontextualizing' instead of 'decontextualizing', as the context, as meaningful situation, is rather changed and thus transformed into a higher level than replaced by context-less abstraction.
[9] We here used the so-called LVS-tests (Janssen, Kraemer and Noteboom, 1995). These tests consist of two parts. One part measures skills in arithmetic with numbers, the second part covers the subjects 'measuring', 'geometry', 'time' and 'money'.
[10] The test items used in the third interview are taken from the 'periodical bearings of the educational level' (PPON) for mathematics (Bokhove, Van der Schoot and Eggen, 1996). In this inquiry in a five year interval the level of mathematics in grade 8 of primary school is observed. The seven items in the interview were taken from the scale 'fractions: applications'.
[11] Note that this final scheme is of a static character in contrast with the scheme's that are adopted from Cook and Campbell (1979), which accentuate a flow of time. However these last scheme's do not offer the opportunity to accurately distinguish between general mathematical skills and curriculum specific skills on the one hand and quantitative and qualitative data on the other.
[12] We will describe and analyse the observations made in the lessons elsewhere (Keijzer & Terwel, in preparation).
[13] As the matching starts with a pre-test, this test serves two objectives. One the one hand it supplied data on the skills of the students at the start of the experiments. On the other hand it proved to be a helpful instrument in a one-to-one matching process of students in experimental group and control group. The pre-test are test from the so-called LVS-series (Janssen, Kraemer and Noteboom, 1995). These tests consist of two parts. The first part measures skills in arithmetic with numbers. The second part covers the subjects 'measuring', 'geometry', 'time' and 'money'. Both parts of the test result in a figure indicating the skill of the student. Moreover as these test are used nation-wide, the measured skills can be used to compare a students' ability with all students in the same grade. Thus a student can be scaled as belonging to one of the categories A, B, C, D or E:

A (p100-75)	B (p75-50)	C (p50-25)	D (p25-10)	E (p10-0)

Students in category A belong to the 25 percent best, those in category B belong to the 25 percent above average and the students in category C belong to the 25 percent under the average. Students in categories D and E are the 25 percent weakest; those in category E are the 10 percent weakest.

[14] Apart from questioning the teachers on general information, we asked the teachers to order the students concerning their skills in mathematics and command of language. We choose these two school subjects, as in combination they characterise school-ability in general. The ranking of the mathematical skills on the other hand formed a means to (re-)interpret the test results. Moreover the command of language is observed as the mathematics education in both experimental group and control group includes the learning from language-rich situations and the learning of a language of fractions, which is anchored in natural language.

[15] Moreover we assume that the mean differences are normally distributed, which too is needed to properly use this test.
[16] Following division of performances of students of the LVS-test (Janssen, Kraemer and Noteboom, 1995), we used to measure the general mathematical skills of the students, we here use the term 'normal performing students', when the students belong to the 75 % best, compared to a national reference for their age-group.
[17] We here found significance levels for the tests on 'numbers and operations' and 'measuring and geometry' of .020 and .024 respectively.
[18] The mean-scores of better performing students and less well performing students on the post-tests in 'numbers and operations' and 'measuring and geometry' show differential effects of the experimental curriculum.

		post-test numbers and operations (control)	post-test numbers and operations (exp)	post-test measuring and geometry (control)	post-test measuring and geometry (exp)
weak students	Mean	61,50	56,75	57,00	54,50
	N	4	4	4	4
	Std. Deviation	3,70	11,06	3,16	16,66
normal performing students	Mean	67,50	71,83	63,00	75,17
	N	6	6	6	6
	Std. Deviation	2,43	5,38	3,52	9,24
Total	Mean	65,10	65,80	60,60	66,90
	N	10	10	10	10
	Std. Deviation	4,18	10,84	4,45	15,93

Again, following division of performances of students of the LVS-test (Janssen, Kraemer and Noteboom, 1995), we used to measure the general mathematical skills of the students, we here use the term 'normal performing students', when the students belong to the 75 % best, compared to a national reference for their age-group.
[19] This finding can be due to lack of statistical power (cf. Cohen, 1988) because of the small number of subjects in this study.
[20] Students from both experimental group and control group were interviewed by one of the researchers, R. Keijzer.
[21] This corresponds with the description from Mcintosh, Reys and Reys (1992) of number sense. They state:
'Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations.' (p. 3)
As in the help offered during the interviews the flexibility in understanding fractions is the main objective, the students are thus stimulated to show ability and inclination to use this understanding in flexible ways to make mathematical judgements. Moreover they are stimulated to develop useful strategies for handling (operations on) fractions.
[22] We again follow the division of performances of students of the LVS-test (Janssen, Kraemer and Noteboom, 1995), we used to measure the general mathematical skills of the students. We here again use the term 'better performing students', when the students belong to the 25 % best, compared to a national reference for their age-group.
[23] Raven tries to let 2/4 operate on a (large) number. Presumably she does not realise that she can take any number as referent to compare 2/4 and ¾. For instance if 60 is the referent, 2/4 and ¾ can be compared by taking 2/4 and ¾ of 60.
It is, however, remarkable that Raven wants to use the operation approach to compare 2/4 and ¾.
[24] We illustrate only the first two points mentioned by Verschaffel. Our research-data cannot provide insight in the development in metacognition of the students in experimental group and control group.
[25] Streefland and Elbers (1995; 1997) frequently emphasised the importance of these class

discussions. Moreover they showed how this interaction in classroom could be established.
[26] In 1987 in the last three years in primary school an average of 14 % of the curriculum in mathematics was used for the teaching of fractions. (Wijnstra,1988).

Appendix 1

Problemsituation from the first interview

In the first interview the students are challenged to formulate the actual query. In the interviews the problems are on separate sheets. On every sheet there is enough space to make drawings or calculations.
The interviewer offers the students standardized help by asking the students:

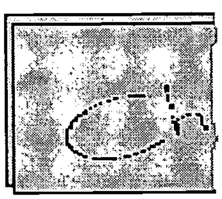
- to explain their solutions;
- to formulate the problem they see, 'What you think is the question here?';
- if they can make a sketch;
- to interpret what they see.

We here present the second problem from the interview.

2. Eating chocolate bars

Irene ate $\frac{3}{5}$ of her chocolate bar.

Here you see what is left.



Comment

In this situation the students read that 3/5 of the bar is eaten. They are asked if this is more than half the bar. This question will eventually motivate them to sketch the full chocolate bar, which can be used to establish what part is not eaten yet.
Next solving the problem can be done by dividing and pacing or trying.

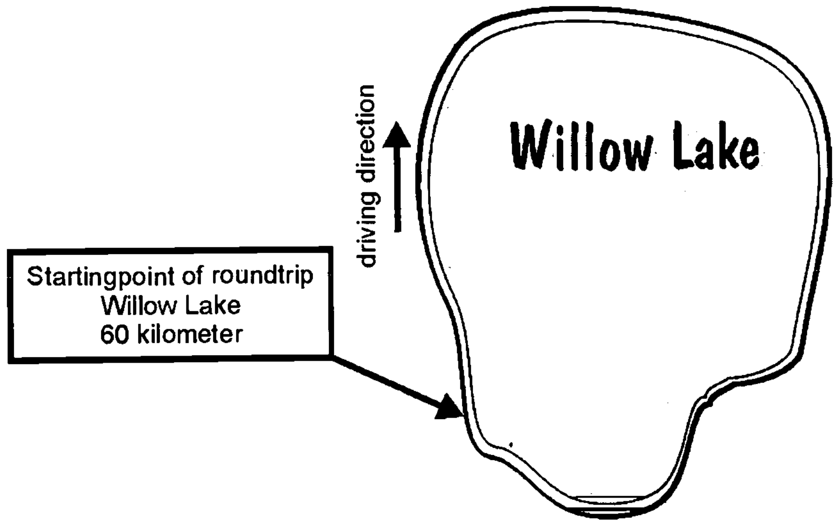
Problemsituations from the second interview

The following two problems were given to the students in the second interview. Like in the first interview the students receive standardized help. As in this case this help is different for each of the presented problems, this is mentioned in the description below.
The second interview aimed at revealing strategies of students to compare fractions.

We here present problem number 1 and problem number 4.

1. Bicycle tour

Mieke, Janneke and Jelle make a bicycle tour around the Willow Lake.



At 12 o'clock Mieke came to $\frac{2}{3}$ and Janneke to $\frac{5}{6}$.

Jelle even came farther.

Comment

The aim of the first problem is to compare 2/3 and 5/6. The (implicit) question namely is who of the children, Mieke or Janneke, came farther at 12 o'clock. The students can compare the fractions by drawing alongside the route, by using the fraction as operator (by calculating how many kilometers were covered by Mieke and Janneke), by comparing the fractions with 1 and by using equivalent fractions (2/3 and 4/6 are equivalent).

The second problem in this situation is to name the distant covered by Jelle in a fraction: 'How far did Jelle come?' The students can solve this problem again by comparing with 1 or to consider fractions equivalent with 5/6.

We constructed the following help for this problem. We ask the students:

- to elucidate the situation: the direction of driving, the length of the tour, distances covered by Mieke, Janneke and Jelle;
- to tell who covered the greatest distance;
- to name the covered distance by Jelle in fractions;
- to use the bicycle path as doublesided number line on which are positioned both fractions and distances;
- to explicate made divisions in the bicycle path.

4. Very close to $\frac{3}{4}$

Make that fraction.

Comment

This task is the most open one. The students can show how they (dare to) construct fractions. Our follow-up question here is: 'Do you know a fraction that is still closer?' In this way we hope to incite the students to reflections on their approaches.

Of course there are many ways to construct fractions near ¾. For instance one can think of the equivalent fraction 6/8 and use this as a basis to appoint 7/8 and 5/8 as nearby fractions. Students, however, could also compare fractions with 1, e.g. 4/5 is situated 1/5 from 1 and ¾

is just ¼ away.
We constructed the following help for this problem. We ask the students:

- how they constructed the fractions near ¾;
- if they could make a fraction even nearer.

Problemsituations from the third interview
The following two problems (from a collection of seven problems) were offered to the students in the third interview. The students first solved these in a written test. In the interview we asked the students to explain their approach. Moreover, if this did not lead to a correct solution the students received standardized help by posing the following questions:

- Can you make a sketch or drawing to show how things work out? Or, could you explain from the picture on the test-sheet?
- A question related to the problem posed (see below).

The third interview aimed at applying knowledge and skills in fractions in rather complex situations.

We here present problem 3 and problem 6.

3. About $\frac{3}{4}$ part of the students of the Plerikschool walks to school. Of the rest half is brought and half comes by bicycle.
What part of the students comes by bicycle?

_____ part

Comment
Here the students need to find a quart as the complement of ¾ and half this quart. Most students will use a (mental) representation of the situation (like a bar or divided circle) to visualize this context. The help offered aids students in structuring the problem in two parts:

- What part of the students does not walk to school?

6. At the beginning of her holidays Janita has *f* 48,- (48 Dutch guilders). $\frac{1}{6}$ part thereof she spends on icecream and limonade. $\frac{1}{4}$ part she spends on picture postcards. The rest she spends on presents.
What part of the money she spends on presents?

_____ part

Comment
To solve this problem actually the number 48 (Dutch guilder) gives superfluous information. The question can be answered by using formal addition and substraction of fractions. However using the number 48 can help to translate the fractions into whole numbers. By interpreting the fraction 1/6 (of) as dividing by 6 it can be established that 48 : 6 = 8 guilders is spent on icecream and limonade. In a similar way the amount spent on picture postcards can be found: 12 guilders. What is left, 28 guilders, needs to be translated into fractions. Therefore students should use a part-whole meaning of fractions in a manner they are not used to.
The specific help that is offered aims at using the number 48 in the situation:

- How much money is spent on icecream and limonade resp. picture postcards?

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